METHOD OF ANALYZING FRONTAL DISCONTINUITIES EXCITED BY A PRESSURE WAVE IN MEMBRANES AND SHELLS

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On the basis of the Laplace transform method, an asymptotic method is proposed for exposing the location and intensity of frontal discontinuities of the solution of a linear second order one-dimensional hyperbolic equation or system of such equations in problems for which the right side of the equation or system has the character of a pressure wave being propagated with decreasing velocity exceeding, at some initial state of the motion, the velocity of wave propagation described by an appropriate homogeneous equation or system of homogeneous equations. At the beginning, the method is elucidated with an example of a second order equation (plane strain of a membrane), and is then generalised for a system of sixth order equations which describes plane and axisymmetric strain wave processes in elastic shells to the accuracy of a Timoshenko-type theory. The method does not take into account the reverse influence of strain on the wave.

Just as in the foundation work of Alumiae [1] devoted to the particular case of a spherical shell subjected to a plane pressure wave, the proposed method utilizes the fact that transverse sections in which the decreasing velocity of the pressure wave becomes equal to the velocity (or one of the velocities) of strain wave propagation, are saddle points for the integrals to be evaluated in the Laplace transform space. The asymptotic of the type in [1] is based sufficiently far from the transverse sections defining the saddle points; the proposed asymptotic, however, is intended primarily for the neighborhood of these transverse sections, and simultaneously yields acceptable results sufficiently far away. There exist particular cases for which the proposed method yields an exact solution for the whole wave process.

The proposed asymptotic formulas have a structure convenient for the construction of particular solutions transposing the strongest discontinuities of the solution, where the remaining part of the solution can be found numerically by a mesh method just as strain wave processes of slabs and shells subjected to edge loading have been studied in [2–5]. Asymptotic formulas of the type in [1] are not suitable for this purpose.

1. Asymptotic method and its application in the one-dimensional wave equation case. Let the prime denote differentiation with respect to the dimensionless coordinate ξ and the dot – with respect to the dimensionless time τ. Let us consider the elucidation of the location and intensity of discontinuities in the solution of the equation

$$u'' - \ddot{u} = q \tag{1.1}$$

under the following side conditions:

a) zero initial conditions are given for $\tau = 0$;

b) a boundary condition or a symmetry condition is given for $\xi = 0$, and the condition $u \to 0$ for $\xi \to \infty$.

c) the function $q(\xi, \tau)$ is symmetric relative to the point $\xi = 0$, and is a pressure-wave type effect whose front (Fig. 1) is given by the equation

$$\tau = p(\xi) \tag{1.2}$$

where $p(\xi)$ satisfies the conditions

$$p(0)=0, \qquad p'(\xi)\begin{cases}<1 & \text{for } 0\leqslant\xi<\xi_1,\\ =1 & \text{for } \xi=\xi_1,\\ >1 & \text{for } \xi>\xi_1,\end{cases} \qquad p''(\xi)>0 \tag{1.3}$$

By virtue of the mentioned conditions it is sufficient to limit the analysis to the domain $\xi\geqslant 0$.

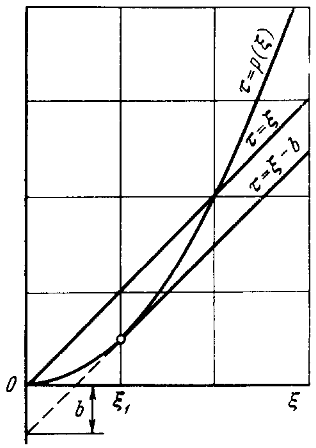

Fig. 1

The formulated mathematical problem can be interpreted physically as a problem to analyze the plane wave process of membrane strain under the effect of a pressure wave or as the problem formulated for a rod in an appropriate way.

Let us define the Laplace transform by means of the formulas

$$F(\xi, s)=\int_0^\infty f(\xi,\tau)e^{-s\tau}d\tau$$

$$f(\xi,\tau)=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}F(\xi,s)e^{s\tau}ds \tag{1.4}$$

Utilizing condition (a), we obtain from (1. 1)

$$U''-s^2U=Q \tag{1.5}$$

We seek the solution of this equation as

$$U=\sum_{j=1}^{2}A_je^{\lambda_j\xi}+\sum_{j=1}^{2}D_jJ_j \qquad (\lambda_1=-s,\ \lambda_2=s) \tag{1.6}$$

$$J_j(\xi)=e^{\lambda_j\xi}I_j(\xi), \qquad I_j(\xi)=\int_0^\xi e^{-\lambda_jx}Q(x)\,dx \tag{1.7}$$

The first sum in (1. 6) represents the general solution of the homogeneous equation (1. 5), and the second sum is the particular solution of the inhomogeneous equation (1.5). Substituting the particular solution, and taking account of the differentiation formulas

$$J_j'=\lambda_jJ_j+Q, \qquad J_j''=\lambda_j^2J_j+\lambda_jQ+Q' \tag{1.8}$$

we have

$$D_1=-D_2=-\frac{1}{2s} \tag{1.9}$$

Utilizing the second of conditions (b), we obtain

$$A_2'(s)=-D_2(s)I_2(\infty) \tag{1.10}$$

Let us examine the calculation of $A_1(s)$ in two cases of the first of conditions (b).

Problem 1. Let $u(0,\tau)=0$, then $U(0,s)=0$, from which follows

$$-A_1(s)=A_2(s)=-\frac{1}{2s}I_2(\infty) \tag{1.11}$$

Problem 2. Let $u'(0,\tau)=0$, then $U'(0,s)=0$, from which follows

$$A_1(s)=A_2(s)=-\frac{1}{2s}I_2(\infty) \tag{1.12}$$

To elucidate the location and intensity of the discontinuities u let us seek the asymptotic approximation of U for $s\to\infty$. As the first step, let us calculate the asymptotic values of the integrals $I_j(\xi)$ as $s\to\infty$. We hence assume that Q has the structure

$$Q = s^{-1/2n}\, G(\xi) e^{-sp(\xi)} \qquad (n=2, 3, 4, \ldots) \tag{1.13}$$

and we formally extend $G(\xi)$, $p(\xi)$ symmetrically in the domain $\xi \leqslant 0$ so that

$$G(-\xi) = G(\xi), \qquad p(-\xi) = p(\xi) \tag{1.14}$$

Let us represent the integrals $I_j(\xi)$ as

$$I_j(\xi) = e^{bs} s^{-1/2n} \int_0^{\xi} G(x)\, e^{-s\varphi_j(x)}\, dx, \quad \varphi_j(\xi) = p(\xi) + \xi\lambda_j s^{-1} + b \geqslant 0 \tag{1.15}$$

Here the constant b is selected so that the condition

$$\varphi_j(\xi_j) = 0 \ \text{(identically for} \ \ j=1,2) \tag{1.16}$$

would be satisfied.

The quantities $\xi_1 = -\xi_2$, $\xi_1 > 0$ are here defined from the conditions

$$\varphi_j'(\xi_j) = 0 \qquad (j=1,2) \tag{1.17}$$

so that

$$\xi = \xi_1, \ \text{if} \quad p'(\xi) = 1; \qquad \xi = \xi_2, \quad \text{if} \quad p'(\xi) = -1 \tag{1.18}$$

The exponential integrands in (1.15) have a stationary value at the saddle points $\xi = \xi_j$, respectively. Hence, to evaluate the $I_j(\xi)$ as $s \to \infty$ it is natural to utilize partially the idea of the method of steepest descent.

Let us represent the integrals (1.15) as the sum of two integrals over intervals containing the saddle point when the saddle point is outside the domain of integration, and let us introduce the new integration variable

$$z = z(x) = \sqrt{s\varphi_j(x)}\, \text{sign}\,(x - \xi_j) \tag{1.19}$$

when $j = 1$, $j = 2$, respectively; then it is easy to give a foundation for the following formula:

$$I_j(\xi) = e^{bs} s^{-\frac{n+1}{2}} \int_{z(0)}^{z(\xi)} \frac{G(\xi)\, e^{-z^2}}{(\sqrt{\varphi_j(\xi)})'}\, dz \qquad (j=1, 2) \tag{1.20}$$

The radicals are here taken with a plus sign

$$\varphi_j(0) = p(0) + b = b \tag{1.21}$$

Utilizing the idea of the method of steepest descent in the integrand of (1.20), we assume

$$G(\xi) \approx G(\xi_j)$$

$$(\sqrt{\varphi_j(\xi)})' \approx (\sqrt{\varphi_j(\xi_j) + (\xi - \xi_j)\varphi_j'(\xi_j) + {}^1/_2(\xi - \xi_j)^2 \varphi_j''(\xi_j)})' = {}^1/_2\sqrt{2p''(\xi_j)} \tag{1.22}$$

We hence have

$$I_j(\xi) \approx N\{\Phi[\sqrt{s\varphi_j(\xi)}\,\text{sign}\,(\xi - \xi_j)] + \Phi[\sqrt{sb}\,\text{sign}\,(\xi_j)]\} \tag{1.23}$$

where $\Phi(x)$ is the probability integral and

$$N = Me^{bs} s^{-1/2(n+1)}, \qquad M \approx G(\xi_1)\sqrt{{}^1/_2\pi / p''(\xi_1)} \tag{1.24}$$

Taking into account the relationships

$$1 - \Phi(x) = \text{erfc}\,(x), \qquad \text{erfc}\,(-x) = 2 - \text{erfc}\,(x)$$

we represent (1.23) in the expanded form

$$\begin{aligned} &I_1(\xi) \approx N\{\text{erfc}[\sqrt{s\varphi_1(\xi)}] - \text{erfc}[\sqrt{sb}]\} \quad \text{for} \ \ 0 \leqslant \xi \leqslant \xi_1 \\ &I_1(\xi) \approx N\{2 - \text{erfc}[\sqrt{s\varphi_1(\xi)}] - \text{erfc}[\sqrt{sb}]\} \quad \text{for} \ \ \xi \geqslant \xi_1 \\ &I_2(\xi) \approx N\{\text{erfc}[\sqrt{sb}] - \text{erfc}[\sqrt{s\varphi_2(\xi)}]\} \quad \text{for} \ \ 0 \leqslant \xi \end{aligned} \tag{1.25}$$

To the same accuracy as the calculations in (1. 11) and (1. 12)

$$I_2(\infty) \approx N\operatorname{erfc}[\sqrt{sb}] \tag{1.26}$$

Taking the above into account, we have the following transform solutions for Problems 1 and 2:

$$U = {}^1/_2 M s^{-1/2(n+2)} \{2Te^{-s\xi} E(sb) - e^{-sp(\xi)}[E(s\varphi_1) + E(s\varphi_2)]\} \quad \text{for } \xi \leqslant \xi_1 \tag{1.27}$$

$$U = {}^1/_2 M s^{-1/2(n+2)} \{2Te^{-s\xi} E(sb) + e^{-sp(\xi)}[E(s\varphi_1) - E(s\varphi_2)] - 2s^{-1/2}e^{(b-\xi)s}\} \quad \text{for } \xi \geqslant \xi_1$$

$$E(sy) = s^{-1/2} e^{sy} \operatorname{erfc}(\sqrt{sy})$$

Here $T = 1$ in the case of Problem 1, and $T = 0$ for the case of Problem 2.

The following inversion formulas as known [6]:

$$e^{-s\gamma} E(sy) \to \frac{H(\tau - \gamma)}{\sqrt{\pi(\tau + y - \gamma)}} \tag{1.28}$$

$$s^{-1/2(m+3)} e^{s(b-\xi)} \to r(\tau - \xi + b)^{1/2(m+1)} H(\tau - \xi + b) \tag{1.29}$$

$$s^{-1/2(m+3)} e^{-s\gamma} E(sy) \to rH(\tau - \gamma) \int_0^{\tau-\gamma} \frac{(\tau - \gamma - t)^{1/2(m+1)}}{\sqrt{\pi(t + y)}}\, dt \tag{1.30}$$

$$r = \frac{1}{[{}^1/_2(m+1)]!} \quad \text{for } m = 1, 3, 5, \ldots$$

$$r = \frac{2^{m+2}}{(m+2)!\sqrt{\pi}} \left[\left(\frac{m+2}{2}\right)\right]! \quad \text{for } m + 2 = 0, 2, 4, \ldots$$

which allow finding discontinuities of the desired solution by means of the asymptotic formulas of (1. 27) as $s \to \infty$.

Note 1.1. In the general case (1. 23)–(1. 27) are approximate because of (1. 22). However, if $G(\xi) = \text{const}$, and $p(\xi)$ is a quadratic polynomial, they will then be exact.

Note 1.2. If $G(\xi)$ and $p(\xi)$ are such that simplifications of (1. 22) become roughly approximate in domains $|\xi - \xi_j| \gg 1$, then it may turn out to be expedient to use the following standard formulas of the method of steepest descent in the domains mentioned [7]:

$$I_1(\xi) = S_1(\xi) \qquad \text{for } 0 < \xi < \xi_1$$

$$I_1(\xi) = S_1(\xi) + \sqrt{2\pi/p''(\xi_1)}\, G(\xi_1) e^{bs} s^{-1/2(n+1)} \quad \text{for } \xi > \xi_1 \tag{1.31}$$

$$I_2(\xi) = S_2(\xi) \qquad \text{for } 0 < \xi$$

$$S_j = \left\{\frac{G(0)}{p'(0) + (-1)^j} - \frac{G(\xi)}{p'(\xi) + (-1)^j} e^{-s[p(\xi) + (-1)^j \xi]}\right\} s^{-1/2(n+2)} \tag{1.32}$$

Similar formulas have been utilized in [1]. Besides (1. 31) and (1. 32) not being applicable in the neighborhood of the point $\xi = \xi_1$, their disadvantage is that they are practically inapplicable for extraction of the particular solution transposing the fundamental discontinuities of the solution because the second member in S_1 will be infinite at $\xi = \xi_1$.

2. Numerical examples for the case admitting an exact solution. Let us consider the case

$$q(\xi, \tau) = H(\tau - k_0\xi^2) \tag{2.1}$$

for which $G(\xi) = 1$, $p(\xi) = k_0\xi^2$ in the notation of Sect. 1, and (1. 27) is the exact transform of the solution of Problems 1, 2. Hence

$$M=\frac{1}{2}\frac{\sqrt{\pi}}{\sqrt{k_0}}, \quad n=2, \quad b=\frac{1}{4k_0}, \quad \varphi_1(\xi)=k_0\left(\xi-\frac{1}{2k_0}\right)^2, \quad \varphi_2(\xi)=k_0\left(\xi+\frac{1}{2k_0}\right)^2$$

Utilizing (1.28)–(1.30) we obtain the exact solution

$$\begin{aligned} u = & -\frac{2}{3\sqrt{k_0}}\tau_1^{3/2}H(\tau_1)H\left(\tau-\frac{1}{4k_0}\right)+T\left[\frac{2}{3\sqrt{k_0}}\tau_1^{3/2}-\frac{1}{2k_0}(\tau-\xi)-\frac{1}{12k_0^2}\right]H(\tau-\xi)+ \\ & +\left\{\frac{1}{3\sqrt{k_0}}\left(\tau_1^{3/2}\left[-1+2H\left(\tau-\frac{1}{4k_0}\right)\right]-\tau_2^{3/2}\right)-\frac{1}{2}\left(\xi-\frac{1}{2k_0}\right)\tau_1+\right. \\ & \left.+\frac{1}{2}\left(\xi+\frac{1}{2k_0}\right)\tau_2+\frac{k_0}{6}\left(\xi-\frac{1}{2k_0}\right)^3-\frac{k_0}{6}\left(\xi+\frac{1}{2k_0}\right)^3\right\}H[\tau-p(\xi)] \\ & \left(\tau_1=\tau-\xi+\frac{1}{4k_0}, \quad \tau_2=\tau+\xi+\frac{1}{4k_0}\right) \end{aligned} \tag{2.2}$$

Here $H(\tau)$ is the Heaviside function, and $T=1$ in the case of the boundary condition $u(0,\tau)=0$ (Problem 1), and $T=0$ in the case of the boundary condition $u'(0,\tau)=0$ (Problem 2).

Differentiation of (2.2) with respect to τ easily yields formulas for the dimensionless velocity and acceleration

$$\begin{aligned} u^{\cdot} = & -\frac{1}{\sqrt{k_0}}\tau_1^{1/2}H(\tau_1)H\left(\tau-\frac{1}{4k_0}\right)+T\left(\frac{1}{\sqrt{k_0}}\tau_1^{1/2}-\frac{1}{2k_0}\right)H(\tau-\xi)+ \\ & +\left\{\frac{1}{2\sqrt{k_0}}\left(\tau_1^{1/2}\left[-1+2H\left(\tau-\frac{1}{4k_0}\right)\right]-\tau_2^{1/2}\right)+\frac{1}{2k_0}\right\}H[\tau-p(\xi)] \end{aligned} \tag{2.3}$$

$$\begin{aligned} u^{\cdot\cdot} = & -\frac{1}{2\sqrt{k_0}}\tau_1^{-1/2}H(\tau_1)H\left(\tau-\frac{1}{4k_0}\right)+T\frac{1}{2\sqrt{k_0}}\tau_1^{-1/2}H(\tau-\xi)+ \\ & +\frac{1}{4\sqrt{k_0}}\left(\tau_1^{-1/2}\left[-1+2H\left(\tau-\frac{1}{4k_0}\right)\right]-\tau_2^{-1/2}\right)H[\tau-p(\xi)] \end{aligned} \tag{2.4}$$

Numerical results for the case $k_0=1$ are presented in Figs. 2 – 4, in which the dashed lines correspond to Problem 1, and the solid lines to Problem 2. Equilevel lines u are

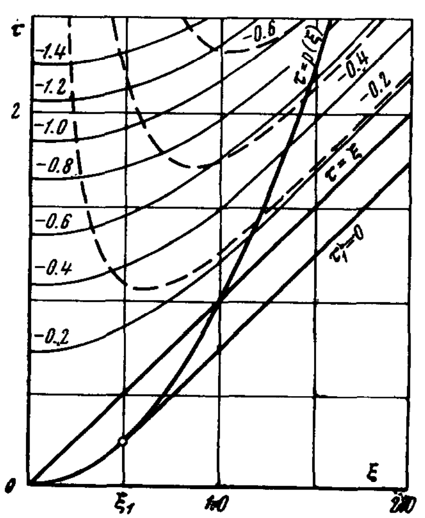

Fig. 2

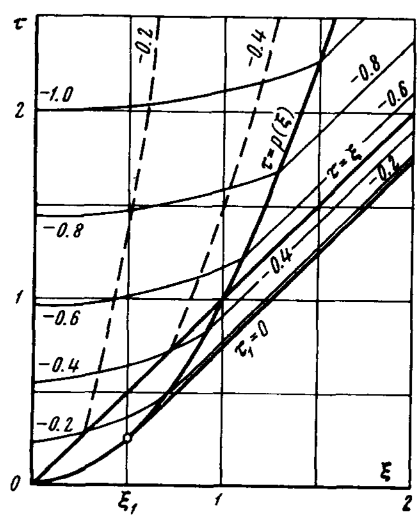

Fig. 3

shown in Fig. 2, and equilevel lines $u^{\cdot}$ in Fig. 3. Shown in Fig. 4 is the change in $u^{\cdot\cdot}$ with respect to ξ for $\tau = 2.5$. In the domain between the fronts $\tau_1 = 0$ and $\tau = \xi$ the values of $u^{\cdot}$ and $u^{\cdot\cdot}$ for Problems 1 and 2 agree.

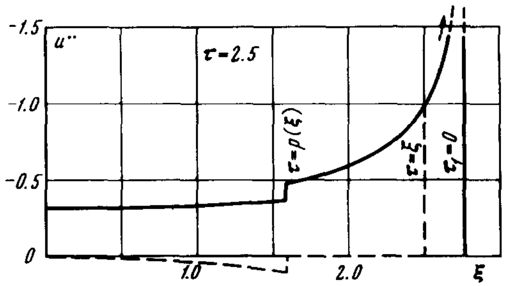

Fig. 4

Let us comment on $u^{\cdot\cdot}$ and the fronts: on the front $\tau = \xi$ we have $u^{\cdot\cdot} = -T$; on the front $\tau = p(\xi)$ the amplitude of $u^{\cdot\cdot}$ increases from $u^{\cdot\cdot}(0) = -1$ to $u^{\cdot\cdot}(1/2k_0) = -\infty$ for $0 \leqslant \xi < 1/2k_0$, and decreases rapidly from the initial value $u^{\cdot\cdot}(1/2k_0) = -\infty$ for $\xi > 1/2k_0$; we have $u^{\cdot\cdot} = -\infty$ on the front $\tau_1 = 0$ starting with the point $\xi = 1/2k_0$.

Using the approximate formulas (1.31) we obtain the normal discontinuities on the fronts $\tau=\xi$ and $\tau_1=0$. However, on the front $\tau=p(\xi)$ the results approximate the normal only for $|\xi - \xi_1| \gg 1$.

The proposed asymptotic method can be generalized to a hyperbolic system of equations with constant coefficients if the roots of the characteristic equation decompose into a pair of different sign. Let us show that it also occurs in the case of shell theory equations having sufficiently slowly changing coefficients.

3. Generalization of the method to a hyperbolic system of equations of axisymmetric and plane strain of shells. On the basis of Timoshenko-type theory, let us consider the axisymmetric and plane wave strain processes of shells of constant thickness $2h$. Let a ξ, η coordinate system with Lamé parameters $A = h$, $B = B(\xi)$ be selected on the middle surface of the shell. Let R_1, R_2 denote the radii of curvature of the middle surface, and let us introduce a dimensionless time by means of the formula $\tau = tc_2/h$, where t is the time, c_2 the velocity of shear wave propagation in elasticity theory.

Let us consider strain processes dependent on ξ, τ. Let u, w be dimensionless (divided by h) displacements in the direction of the coordinate ξ and normal to the middle surface, respectively, ψ the angle of rotation of the normal, k_*^2 the shear coefficient, ν Poisson's ratio, $q_0(\xi, \tau)$ the dimensionless normal pressure. Let us introduce the following notation

$$k^2 = (1-\nu)/2, \quad K = k^2 k_*^2 \tag{3.1}$$

$$P = \frac{h}{R_1} + \frac{\nu h}{R_2}, \qquad P_1^2 = \frac{h^2}{R_1^2} + \frac{2\nu h^2}{R_1 R_2} + \frac{h^2}{R_2^2} \tag{3.2}$$

Let the primes denote derivatives with respect to ξ, and dots with respect to τ.

Let us take the following modification of equations of a Timoshenko-type theory as basis: (*)

$$u'' + \frac{B'}{B}u' + \left[\nu\frac{B''}{B} - \left(\frac{B'}{B}\right)^2\right]u - k^2 u^{\cdot\cdot} + Pw' = 0 \tag{3.3}$$

$$\frac{1}{3}\psi'' + \frac{1}{3}\frac{B'}{B}\psi' + \left[\frac{1}{3}\nu\frac{B''}{B} - \frac{1}{3}\left(\frac{B'}{B}\right)^2 - K\right]\psi - \frac{1}{3}k^2\psi^{\cdot\cdot} - Kw' = 0$$

*) At the end of the section it will be explained in what sense it is sufficient to take the simplified system (3.3) as the initial system.

$$Pu' - K\left(\psi' + \frac{B'}{B}\psi\right) - Kw'' - K\frac{B'}{B}w' + P_1^2 w + k^2 w^{\cdot\cdot} = q_0$$

Depending on the choice of coordinates (3.3) permits description of axisymmetric or plane wave processes of shell deformation. In the particular case of a slab $R_1 = R_2 = \infty$, $P = 0$ and, therefore, the first equation determines u separately (symmetric strain), and the system of the second and third equations determines ψ and w (bending strain).

In the domain $\xi \geqslant 0$ let us investigate discontinuities in the wave solution of the system (3.3), constructed under the following side conditions:

a) zero initial conditions are given for $\tau = 0$;

b) for $\xi = 0$ three homogeneous boundary conditions or symmetry conditions are given, and at infinity the conditions

$$u(\xi, \tau) \to 0, \quad \psi(\xi, \tau) \to 0, \quad w(\xi, \tau) \to 0, \qquad \text{for } \xi \to \infty \tag{3.4}$$

c) the function $q_0(\xi, \tau)$ is an effect of pressure-wave type whose front is given by (1.2), where $p(\xi)$ satisfies conditions (1.3) with the sole difference that as ξ grows we demand a smooth increase in $p'(\xi)$ from the value $p'(0) < k$ to a value exceeding the quantity $1/k_*$.

Taking (a) into account, let us take the Laplace transform (1.4) of the system (3.3) by denoting the transforms of u, ψ, w, q_0 by U, Ψ, W, Q_0, respectively. Now if we introduce the new functions

$$V_1 = U\sqrt{B}, \qquad V_2 = \Psi\sqrt{B}, \qquad V_3 = W\sqrt{B}, \qquad Q = Q_0\sqrt{B} \tag{3.5}$$

we obtain the following system of ordinary differential equations

$$V_1'' - [3B_1^2 + (1-2\nu)B_2 + k^2s^2]V_1 + P(V_3' - B_1V_3) = 0 \tag{3.6}$$

$$\tfrac{1}{3}V_2'' - \tfrac{1}{3}[3B_1^2 + (1-2\nu)B_2 + k^2s^2 + 3K]V_2 - K(V_3' - B_1V_3) = 0$$

$$P(V_1' - B_1V_1) - K(V_2' + B_1V_2) - K[V_3'' - (-B_1^2 + B_2 + P_1^2K^{-1} + k^2s^2K^{-1})V_3] = Q$$

$$B_1 = B' / 2B, \; B_2 = B'' / 2B \tag{3.7}$$

Let us consider the construction of the asymptotic solution of the system (3.6) as $s \to \infty$, where the original determines the discontinuity of the solution of the original system (3.3). We hence assume that the following simplifying conditions are satisfied in the shell domain under consideration:

1) $B \neq 0$

2) $1 \gg \vartheta^2, \quad \vartheta^2 \gg B_1^2, \; |B_2|, \; h^2/3R_1^2, \; h^2/3R_2^2$

3) The radii of curvature R_1, R_2 are either constant or slowly changing functions of ξ, so that it is sufficient to consider R_1, R_2 as constants in constructing the correcting elements of the main terms of the asymptotic solution as $s \to \infty$. If the estimate $\vartheta_0 \gg \xi P'/P, \; \xi P''/P^2, \; P'/P^2$ is introduced this condition is satisfied for $\vartheta_0 \ll 1$. In the case $\vartheta_0 \sim 1$ the correcting elements to the principal terms of the asymptotic solution containing P have the character of the estimate.

Let us seek the V_i as

$$V_i = V_{i0} + V_{i1} \qquad (i = 1, 2, 3) \tag{3.8}$$

where V_{i0} is the asymptotic general solution of the homogeneous system (3.6) and V_{i1} is the asymptotic particular solution of the inhomogeneous system (3.6).

Direct substitution in the homogeneous system (3.6) with the mentioned assumptions taken into account, easily verifies that the functions V_{i0} can be selected in the form

$$V_{i0} = \sum_{j=1}^{6} A_j C_{ij} e^{\lambda_j \xi} \qquad (i = 1, 2, 3) \tag{3.9}$$

Here $\lambda_j (s)$ is in a second approximation (3.10)

$$\lambda_1 = -ks - \frac{P^2}{2ks}, \quad \lambda_2 = -ks + \frac{3K}{2(K-1)ks}, \quad \lambda_3 = -\frac{s}{k_*} - \frac{3Kk_*}{2(K-1)s}$$

$$\lambda_4 = -\lambda_1, \qquad \lambda_5 = -\lambda_2, \qquad \lambda_6 = -\lambda_3 \tag{3.11}$$

The coefficients A_j (s) should satisfy conditions (b), and the C_{ij} are determined in a first approximation as

$$C_{11} = 1, \qquad C_{21} = P, \qquad C_{31} = P / ks$$

$$C_{12} = -P / 3, \quad C_{22} = 1, \quad C_{32} = K / (K-1)\, ks \tag{3.12}$$

$$C_{13} = -k_* P / (K-1)s, \quad C_{23} = 3k_* K/(K-1)s, \quad C_{33} = 1$$

$$C_{1,j+3} = C_{1j}, \quad C_{2,j+3} = C_{2j}, \quad C_{3,j+3} = -C_{3j} \qquad (j = 1, 2, 3) \tag{3.13}$$

The coefficients of powers of s having the order of unity in (3.10), (3.12) have been constructed with an error on the order of ϑ^2, and the coefficients having the order P or P^2 are roughly approximate for the variables B, R_1 and R_2. However, this fact does not introduce any essential error because of our assumption $P \lesssim \vartheta \ll 1$.

Note 3.1. Let us consider the domain $\xi \geqslant 0$. To do this all members of the sum (3.9) are needed. If the origin of waves in the domain $\xi \geqslant 0$ as a result of the edge effect at $\xi = 0$ is considered, then we have $A_4 = A_5 = A_6 = 0$ from (3.4), and only the first three members of the sum (3.9) which contains coefficients allowing satisfaction of the three boundary conditions given at $\xi = 0$ can be taken into account.

Utilizing the idea of the method of variation of constants, let us seek V_{i1} in the form

$$V_{i1} = \sum_{j=1}^{6} D_j C_{ij} J_j \qquad (i = 1, 2, 3) \tag{3.14}$$

where the J_j are defined by (1.7), and the D_j (s) are the desired coefficients. In differentiating V_{i1} we shall, on the basis of the above-mentioned assumptions consider those elements λ_j, C_{ij} which contain P as constants, and we shall utilize (1.8).

Substituting V_{i1} in the form (3.14) into (3.6), and differentiating, we easily see that in all three equations the coefficients of J_j vanish to the same accuracy as the individual members of the sum (3.9) satisfy the homogeneous system (3.6). By equating the coefficients of Q and Q' to zero in all three equations, we obtain a system of equations which can be reduced to the form

$$\sum_{j=1}^{6} D_j C_{ij} = 0, \quad \sum_{j=1}^{6} D_j \lambda_j C_{ij} = \begin{cases} 0 & \text{for } i = 1, 2 \\ -K^{-1} & \text{for } i = 3 \end{cases} \qquad (i = 1, 2, 3) \tag{3.15}$$

If we take $D_i + D_{i+3}$, $D_i - D_{i+3}$ $(i = 1, 2, 3)$ as the new desired quantities, we easily establish that

$$D_4 = D_1, \qquad D_5 = D_2, \qquad D_6 = D_3 \tag{3.16}$$

and we deduce the equations

$$C_{i1} D_1 + C_{i2} D_2 + C_{i3} D_3 = 0 \quad (i = 1, 2) \tag{3.17}$$
$$C_{31}\lambda_1 D_1 + C_{32}\lambda_2 D_2 + C_{33}\lambda_3 D_3 = -1 / 2K$$

for the evaluation of D_1, D_2, D_3.

Utilizing (3.10), (3.12), we easily construct the asymptotic solution of (3.17) for $s \to \infty$ in the form

$$D_1 = -\frac{P}{2k^2 s^2}, \qquad D_2 = -\frac{3K}{2(K-1)k^2 s^2}, \qquad D_3 = \frac{k_*}{2Ks} \tag{3.18}$$

On the basis of the above, we have

$$V_i = \sum_{j=1}^{6} C_{ij} [A_j + D_j I_j(\xi)] e^{\lambda_j \xi} \qquad (i = 1, 2, 3) \tag{3.19}$$

Here the $I_j(\xi)$ are determined from (1.7), and the $A_j(s)$ are still arbitrary coefficients to be evaluated from condition (b). Using this latter in part of (3.4) we have

$$A_l = -D_l I_l(\infty) \quad (l = 4, 5, 6) \tag{3.20}$$

Taking account of (3.11), (3.13), this permits rewriting (3.19) as

$$V_i = \sum_{j=1}^{3} C_{ij} \{ [A_j + D_j I_j(\xi)] e^{\lambda_j \xi} + r_i D_j [I_{j+3}(\xi) - I_{j+3}(\infty)] e^{-\lambda_j \xi} \}$$
$$(r_1 = 1, \ r_2 = 1, \ r_3 = -1) \tag{3.21}$$

where $A_1(s)$, $A_2(s)$, $A_3(s)$ are coefficients determined from the conditions at $\xi = 0$. Let us examine their evaluation in two cases.

Problem 1. Given the conditions

$$u(0, \tau) = 0, \qquad \psi(0, \tau) = 0, \qquad w(0, \tau) = 0 \tag{3.22}$$

from which follow the conditions $V_i(0, s) = 0$ allowing for the formation of a system of three equations in A_1, A_2, A_3 on the basis of (3.21). The asymptotic solution of this system as $s \to \infty$ is the following:

$$A_1 = D_1 I_4(\infty) - \frac{2Pk_* K}{(K-1)^3 ks^2} D_2 I_5(\infty) + \frac{2Pk_*}{s} D_3 I_6(\infty)$$
$$A_2 = \frac{6Pk_* K}{(K-1) ks^2} D_1 I_4(\infty) + D_2 I_5(\infty) + \frac{6k_* K}{(K-1)s} D_3 I_6(\infty) \tag{3.23}$$
$$A_3 = -\frac{2P}{ks} D_1 I_4(\infty) - \frac{2K}{(K-1) ks} D_2 I_5(\infty) - D_3 I_6(\infty)$$

Problem 2. Given the symmetry conditions

$$u(0, \tau) = 0, \qquad \psi(0, \tau) = 0, \qquad w'(0, \tau) = 0, \qquad B'(0) = 0 \tag{3.24}$$

from which follow the conditions $V_1(0, s) = 0$, $V_2(0, s) = 0$, $V_3'(0, s) = 0$ allowing the formation of a system of three equations on the basis of (3.21) and (1.8). with the solution

$$A_j = D_j I_{j+3}(\infty) \qquad (j = 1,2,3) \tag{3.25}$$

Furthermore, let $T = 1$ for Problem 1, and $T = 0$ for Problem 2. Then, on the basis of the above, single expanded formulas of the asymptotic transforms of the solutions as $s \to \infty$ can be constructed for these problems

$$U = P\vartheta_2 \{ -(K-1) \ [I_4(\infty) - 2TI_6(\infty) + I_1(\xi)] e^{\lambda_1 \xi} +$$
$$+ (K-1) \ [I_4(\infty) - I_4(\xi)] e^{-\lambda_1 \xi} + K \ [I_5(\infty) - 2TI_6(\infty) + I_2(\xi)] e^{\lambda_2 \xi} -$$

$$- K[I_5(\infty) - I_5(\xi)]e^{-\lambda_2\xi} - [(1-2T)I_6(\infty) + I_3(\xi)]e^{\lambda_3\xi} + \\ + [I_6(\infty) - I_6(\xi)]e^{-\lambda_3\xi}\}$$

$$\Psi = \vartheta_2\{-P^2(K-1)[I_4(\infty) + I_1(\xi)]e^{\lambda_1\xi} + P^2(K-1)\ [I_4(\infty) - \\ - I_4(\xi)]e^{-\lambda_1\xi} - 3K\ [I_5(\infty) - 2TI_6(\infty) + I_2(\xi)]e^{\lambda_2\xi} + \\ + 3K\ [I_5(\infty) - I_5(\xi)]e^{-\lambda_2\xi} + 3K[(1-2T)I_6(\infty) + I_3(\xi)]e^{\lambda_3\xi} - \\ - 3K\ [I_6(\infty) - I_6(\xi)]e^{-\lambda_3\xi}\} \qquad (3.26)$$

$$W = \vartheta_1\{-P^2(K-1)^2[I_4(\infty) + I_1(\xi)]e^{\lambda_1\xi} - P^2(K-1)^2\ [I_4(\infty) - \\ - I_4(\xi)]e^{-\lambda_1\xi} - 3K^2[I_5(\infty) + I_2(\xi)]e^{\lambda_2\xi} - 3K^2[I_5(\infty) - I_5(\xi)]e^{-\lambda_2\xi} + \\ + T\ [2P^2(K-1)^2 I_4(\infty) + 6K^2 I_5(\infty)e^{\lambda_3\xi}]\} + \theta\{[(1-2T)I_6(\infty) + \\ + I_3(\xi)]e^{\lambda_3\xi} + [I_6(\infty) - I_6(\xi)]e^{-\lambda_3\xi}\}$$

$$\vartheta_1^{-1} = 2(K-1)^2\sqrt{B}k^3s^3, \quad \vartheta_2^{-1} = 2(K-1)\ \sqrt{B}k^2s^2, \quad \theta = k_* / 2K\sqrt{B}s$$

For Problem 2 formulas (3.26) have been obtained on the basis of (3.5), (3.12), (3.18), (3.21), (3.25) without any simplification, and for Problem 1, on the basis of (3.5), (3.10), (3.12), (3.18), (3.21), (3.23) with only the factor containing the highest power of s being retained in each $I_j(\xi)$ and $I_j(\infty)$ $(j = 1, 2, \ldots, 6)$ and with the admission of an error in the coefficients on the order of P^2. It is easy to verify that the transforms (3.26) satisfy the boundary conditions and conditions (3.4) exactly. However, they satisfy (3.6) in the asymptotic sense as $s \to \infty$ with an error in the coefficients on the order of ϑ^2.

Inversion of the transforms (3.26) can be performed with different accuracy. However, we limit ourselves herein to the application of the method proposed in Sect. 1.

Let us introduce the notation $s_1 = ks, \quad s_3 = s / k_*$ (3.27)

Then upon using (3.10) in a first approximation (3.28)

$$\lambda_{1,2} = \lambda_1^{(1)} = -s_1, \quad \lambda_{4,5} = \lambda_2^{(1)} = s_1, \quad \lambda_3 = \lambda_1^{(3)} = -s_3, \quad \lambda_6 = \lambda_2^{(3)} = s_3$$

To the accuracy of (3.28)

$$I_{1,2} = I_1^{(1)}, \qquad I_{4,5} = I_2^{(1)}, \qquad I_3 = I_1^{(3)}, \qquad I_6 = I_2^{(3)} \qquad (3.29)$$

where $I_j^{(l)}(\xi)$ $(j = 1, 2;\ l = 1, 3)$ are determined by (1.7) for $I_j(\xi)$ $(j = 1, 2)$ by selecting $\lambda_j^{(l)}$ $(l = 1, 3)$, respectively in place of λ_j.

Using (3.27), (3.29), and admitting an error of order P^2 in the coefficients, we obtain the following simplified formulas from (3.26):

$$U = U^{(1)} + U^{(3)}, \qquad \Psi = \Psi^{(1)} + \Psi^{(3)}, \qquad W = W^{(1)} + W^{(3)} \qquad (3.30)$$

$$U^{(l)} = \frac{P}{2(K-1)\sqrt{B}} s_1^{-2} F_l, \quad \Psi^{(l)} = -\frac{3K}{2(K-1)\sqrt{B}} s_1^{-2} F_l \quad (l = 1, 3) \qquad (3.31)$$

$$F_1 = [I_1^{(1)}(\xi) + I_2^{(1)}(\infty) - 2TI_2^{(3)}(\infty)]e^{-s_1\xi} + [I_2^{(1)}(\xi) - I_2^{(1)}(\infty)]e^{s_1\xi}$$

$$F_3 = -[I_1^{(3)}(\xi) + (1-2T)I_2^{(3)}(\infty)]e^{-s_3\xi} - [I_2^{(3)}(\xi) - I_2^{(3)}(\infty)]e^{s_3\xi} \qquad (3.32)$$

$$W^{(1)} = -\frac{3K^2}{2(K-1)^2\sqrt{B}} s_1^{-3}\{[I_1^{(1)}(\xi) + I_2^{(1)}(\infty)]e^{-s_1\xi} - \\ - [I_2^{(1)}(\xi) - I_2^{(1)}(\infty)]e^{s_1\xi} - 2TI_2^{(1)}(\infty)e^{-s_3\xi}\}$$

$$W^{(3)} = \frac{1}{2K\sqrt{B}} s_3^{-1} \{ [I_1^{(3)}(\xi) + (1-2T) I_2^{(3)}(\infty)] e^{-s_3\xi} - - [I_2^{(3)}(\xi) - I_2^{(3)}(\infty)] e^{s_3\xi} \} \tag{3.33}$$

As in Sect. 1, we assume that Q has the structure of (1. 13). Now if we introduce the notation

$$p_1(\xi) = p(\xi) / k, \qquad p_3(\xi) = p(\xi) k_* \tag{3.34}$$

then (1. 15)–(1. 26) can be used to evaluate the $I_j^{(l)}(\xi)$ $(j = 1,2;\ l = 1,3)$ in (3. 22), (3. 33) by replacing the quantities

$$I_j,\ \varphi_j,\ \lambda_j,\ \xi_j,\ s,\ p,\ b,\ M, N,\ G(\xi_j)$$

respectively, by quantities

$$I_j^{(l)},\ \varphi_j^{(l)},\ \lambda_j^{(l)},\ \xi_j^{(l)},\ s_l,\ p_l,\ b^{(l)},\ M^{(l)},\ N^{(l)},\ G(\xi_j^{(l)})$$

For two different values of the superscript l different pairs $(j = 1,2)$ of saddle points $\xi = \xi_j^{(l)}$ will substantially be applied.

Thus, representing $I_j^{(l)}$ $(j = 1,2;\ l = 1,3)$ in (3. 32), (3. 33) as formulas of type (1. 25), we obtain for expressions for U, Ψ, W from (3. 30) which may be inverted by applying (1. 28)–(1. 30) for specifically given $n = 2, 3, 4, \ldots$ The following circumstance should hence be kept in mind. Firstly, in conformity with (3. 5), $Q = Q_0\sqrt{B}$ in (1. 13). Secondly, when s is replaced by s_l in the left sides of (1. 28)–(1. 30), the τ should be replaced on the right side by $\tau^{(l)}$, where

$$\tau^{(1)} = \tau / k, \qquad \tau^{(3)} = \tau k_*$$

and the factors $1/k$ and k_* inserted, respectively.

Note 3.2. There is a factor s_l^{-2} in (3. 30) for $U^{(l)}$, $\Psi^{(l)}$ $(l = 1, 3)$, hence formulas (3. 33) for $W^{(1)}, W^{(3)}$ correspondingly contain the factors s_1^{-3} and s_3^{-1}. Therefore, independently of the specific nature of the pressure wave, the discontinuity corresponding to $W^{(3)}$ is most essential. Let us call this discontinuity provisionally a first order discontinuity. Then $U^{(l)}$, $\Psi^{(l)}$ $(l = 1, 3)$ define the second order discontinuities, and $W^{(1)}$ the third order discontinuity. Here $U^{(1)}$, $\Psi^{(1)}$, $W^{(1)}$ define the discontinuities in u, ψ, w on fronts being propagated with the dimensionless velocity $1/k$, and $U^{(3)}$, $\Psi^{(3)}$, $W^{(3)}$ on fronts being propagated with the dimensionless velocity k_*. The discontinuities on the front $\tau = p(\xi)$ are determined by the total contribution of $U^{(l)}$, $\Psi^{(l)}$, $W^{(l)}$ $(l = 1, 3)$ where the contribution of the components $l = 1$ will dominate in the neighborhood of the point $\xi = \xi_1^{(1)}$, and the contributions of the components $l = 3$ in the neighborhood of the point $\xi = \xi_1^{(3)}$.

The simplified system (3. 3) was taken as the original system above to shorten the exposition.

Let us now assume that a more exact system of equations of motion of a Timoshenko-type theory is derived from the standard equilibrium equations in stress resultants, moments, and transverse forces by the following means: (a) the stress resultants and moments are expressed in terms of displacements by using the simplest Novozhilov-Balabukh elasticity relationships; (b) the transverse force is given as $2hEK(1-\nu^2)^{-1}(\psi - uh/ / R_1 + w')$, (c) inertial terms are introduced without taking account of the change in metric in the shell thickness.

It turns out that the asymptotic solution as $s \to \infty$ of the appropriate Laplace transformed system of equations agrees with (3. 26) in the part of the element W having the factor s^{-1} and differs from (3. 26) most substantially only in corrections on the order of

h/R_1, h/R_2 (as compared with unity) to the coefficients of elements in (3.26) having a factor of order s^{-2}.

4. Simplified method of construction of the asymptotic solution of the transformed shell equations. The system (3.6) has specific properties allowing the construction of its asymptotic solution (3.30)–(3.33) as $s \to \infty$ by a simpler method than the general method described in the preceding section.

Using the notation (3.27), we introduce into the considerations a simplified system (3.6) in the form

$$L_1V_1 = -PV_3', \quad L_1V_2 = 3KV_3', \quad K^{-1}PV_1' - V_2' - L_3V_3 = -Q^* \tag{4.1}$$

$$L_l = \frac{\partial^2}{\partial \xi^2} - s_l^2 \quad (l = 1,3), \qquad Q^* = -\frac{Q}{K} \tag{4.2}$$

On the basis of (4.1), an asymptotic solution of (3.30)–(3.33) of the original system (3.6) as $s \to \infty$ can be constructed for Problems 1 and 2 by means of the following steps.

1) By integrating the equation

$$L_3V_3 = Q^* \tag{4.3}$$

which does not differ substantially from (1.1), taking account of the condition $V_3(0, s) = 0$ in the case of Problem 1 or the condition $V_3'(0, s) = 0$ in the case of Problem 2, we obtain $W^{(3)}\sqrt{B}$ as solution.

2) By constructing the solution of the first two equations of the system (4.1), using their now known right sides and taking account of the boundary conditions $V_1(0, s) = 0$, $V_2(0, s) = 0$ and the conditions at infinity, we obtain $U\sqrt{B}$ and $\Psi\sqrt{B}$.

3) By constructing a particular solution of the equation

$$L_3V_3 = K^{-1}PV_1' - V_2' \tag{4.4}$$

a formula for $W^{(1)}\sqrt{B}$ with error on the order of P^2 can now be obtained by using the known V_1 and V_2.

The method mentioned remains valid for a broad class of other boundary conditions. It will also be effective in the investigation of discontinuities taking account of the influence of the surrounding medium, because only (4.3) supplemented by a term taking account of the influence of the medium should be integrated in combination with the equation of the medium.

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APPLYING THE EXPLOSION ANALOGY TO THE CALCULATION OF HYPERSONIC FLOWS

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After Tsien [1], Hayes [2] and Il'iushin [3] had established the analogy between hypersonic flow past slender bodies,and unsteady flows in a space with one fewer dimensions, many researchers sought to establish which steady flow corresponds to the motion of a gas produced by an intense explosion. The authors of the earliest studies [4–8] assumed that the gas particles in an explosion of a flat or filament charge move in the same way as in flow near a blunt plate or semi-infinite cylinder at a zero angle of attack relative to the free stream. The thickness of the streamlined bodies were assumed to be infinitesimal; the bluntness of their leading edges was taken to be the direct analog of the action of a concentrated force on the ambient medium. The resulting analogy made it possible to isolate the most salient common features of the two effects, but suffered from one drawback: the density at the plate and cylinder surfaces turned out to equal zero, and the entropy to be infinite.

Cheng [9], Sychev [10, 11] and Yakura [12] subsequently developed the notion of a high-entropy layer whereby the thickness of streamlined bodies increases to infinity downstream, while the entropy remains finite over the entire contour. They emphasized that flow in a high-entropy layer differs from that in the rest of space, and that the use of the hypothesis of plane cross sections to calculate this layer entails considerable errors.

The results of Sychev [10, 11] and Yakura [12] are thoroughly analyzed below. It is shown that these results are obtainable directly from the theory of intense explosions as developed by Sedov [13, 14] and Taylor [15]. This possibility means that the analogy between unsteady flows and hypersonic flow past slender bodies is valid in the first approximation throughout the domain beyond the front of the bow shock wave. This includes the domain adjacent to the contour of the body. The contour itself can be determined simply by choosing an appropriate value of the entropy at the particle trajectory which generates it; the equation of the trajectory can be found by solving the explosion problems in Lagrange variables [16].

1. We assume that the motion of the gas is axially symmetric. Our principal conclusions will be equally valid for plane-parallel flows, however. We denote the axes of the cylindrical coordinate system by x and r, directing the x-axis along the velocity vector of the unperturbed stream. Following [10–12], we shall consider the inverse problem, i. e. we shall prescribe the form of the shock wave $r = r_s(x)$, and determine the contour of the streamlined body in the course of solution. Using the explosion analogy to